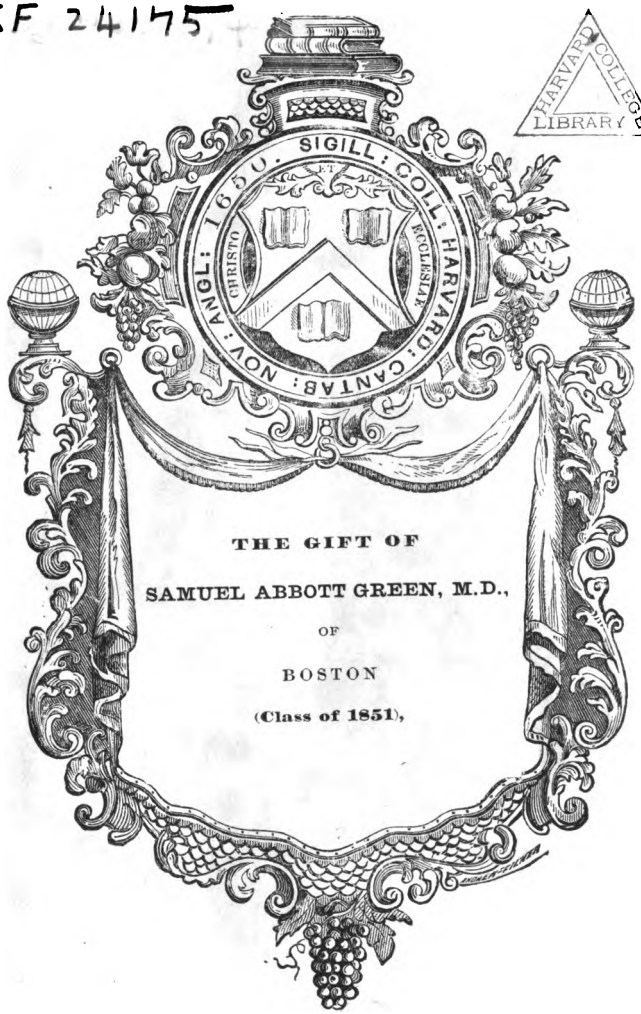

EXPLANATION

OF THE PRINCIPLES OF

CRYSTALLOGRAPHY AND CRYSTALLOPHYSICS.

BY

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TRANSLATED FOR THE SMITHSONIAN INSTITUTION
BY PROF. T. EGLESTON.

REPRINTED FROM THE SMITHSONIAN REPORT
FOR 1872.

WASHINGTON:
GOVERNMENT PRINTING OFFICE.
1874.

EXPLANATION OF THE PRINCIPLES OF CRYSTALLOGRAPHY AND CRYSTALLOPHYSICS.[1]

By Aristides Brezina.

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INTRODUCTION.

Among all the methods in crystallography there is not a single one which has remained so completely confied within special limits as Miller's. The reason for this is not the abstract method in which the subject is treated, nor its difficult mathematical principles, but is principally owing to the fact that, up to this time, it has never been treated separately from those operations which serve for the derivation of certain special mathematical formulæ in the first principles of geometry.

Miller's method is really capable of an elementary treatment, which, almost without the use of mathematics, renders possible not only the quick and certain explanation of all combinations in the way of zone-observations, but also the recognition of the physical characters of crystals on the basis of their relations of symmetry.

These characters of this known method are especially useful for mineralogists and lithologists, who make microscopical observations: for the first, because he, without many measurements and calculations, can show, from only the simple inspection of a crystal, the connection of the different faces, together with the explanation of the combination; and for the latter, because he is in position, on account of a precise knowledge of the relations of symmetry, to recognize, in thin sections, both the crystalline system and the elements of a crystal; and in both cases, without presumption of such mathematical knowledge, which is without the departments of mineralogists and lithologists. This method is, for this reason, not only simple and fundamental, but is in every way superior to the others in use, which have originated with Weiss, Naumann, and Levy.

One of the most important advantages of it is the possibility of a simultaneous development of the crystallographic and physical relations of every system from its known symmetry. This method of procedure gives, from the very commencement, a complete insight into its habits and characteristics, and secures, during its development, a survey of the whole theoretical structure. But while this method of derivation was first carried out for the crystallographic part by von Lang[2], the in-

[1] Mineralogische Mittheilungen. Wien, 1872.

[2] Lang, Krystallographie. Wien, Braumüller, 1866.

troduction of Whewell's[1] method of notation of the faces of crystals was an important element in Miller's system.

Miller's symbols consist, as will be explained later, of three indices, which are inversely proportional to the intersections of the faces on the three axes; while in Weiss's system they are directly proportional. Naumann's and Levy's systems sometimes give sections of the axes and sometimes the relations betweeen two sections. The advantages of Miller's notations are very numerous: Chiefly it allows of representing every individual face; while Naumann's and Levy's symbols give only the form, *i. e.*, the re-union of all the faces which belong together. When it is necessary to represent the whole form in Miller's system, the symbol of the face is represented in parentheses; it has, therefore, the advantage, that, according as it is required, either the face or the form can be exactly and concisely designated.

Miller's symbol is, besides, exceeding simple and convenient. While here three low whole numbers, 0, 1, and seldom 2, are sufficient, in Weiss's system three or four fractions, and three or four letters, in groups of three or four, are required, separated by colons:

$$\left|\tfrac{1}{2}a : b : \infty c\right| \text{ or } \left|\tfrac{1}{2}a' : a' : 2a' : c\right|$$

In Naumann's, two fractions and a letter, with perhaps as many as four accents, as:

$$2\breve{P}\infty \text{ or } \tfrac{1}{3}\breve{P}'_{/} : 2$$

Levy's symbols are, in many cases, complicated, as in pyramids:

$$b^{\frac{1}{2}}\, d^{1}\, d^{\frac{1}{2}}$$

where there are three letters and three fractions.

Naumann's, and Levy's symbols are not symmetrical with regard to the crystallographic axes, *i. e.*, while with Miller the first, second, and third indexes refer invariably to the first, second, and third axes, it is never the case with Naumann, and with Levy only in the most complicated cases (the pyramid of the second order) that every axis is represented by an index, and even in this case the signs of the axes change their position. This symmetry of axes is important, because it makes both the transformation of the indices in changes of axes, as well as the calculation of zone-equations, exceedingly simple and demonstrative.

Singularly enough, this side of Miller's symbols has been attacked because in Naumann's and Levy's symbols the difference between pyramids, prisms, domes, and pinacoids is apparent. This is, however, extremely unjust. In Miller's system, in the symbol of the pyramid, there are three 0's of different values. In the symbol of a prism or dome, an index is equal to 0; a pinacoid has the symbol (1 0 0), (0 1 0), or (0 0 1), which contains two 0's, and is certainly a difference which strikes the eye.

As opposed to the notation of Weiss, Miller's method, besides the

[1] Whewell, Phil. Trans., 1825, p. 87.

brevity mentioned above, has the further advantage that, instead of the symbol ∞, zero is used, because the figures of both these systems are reciprocal. How great the importance of this particular is in the calculation of zone-equations will be immediately shown. On the facility of zone-development, however, depends the quick and sure solution of the combination.

The method of establishing a zone-equation is, according to Miller, as follows: Given two faces, $e\,f\,g$ and $p\,q\,r$, the sign of the zone formed by both can be obtained by crosswise multiplication and subtraction, as follows:

$$\begin{array}{c} e\,f\,g \quad e\,f\,g \\ \times\times\times \\ p\,q\,r \quad p\,q\,r \\ \hline [fr-gq;\ gp-er;\ eq-fp] \\ [u \qquad v \qquad w] \end{array}$$

$[u\,v\,w]$ is the symbol of the zone; now, $e\,f\,g$ $p\,q\,r$ are severally whole numbers; the products, $f\,r$, $g\,q$, $g\,p$, ..., are, for that reason, likewise so; the same is therefore true of their differences, which represent the indices $u\,v\,w$ of the zone.

If the face $x\,y\,z$ lies in the zone represented by $[u\,v\,w]$, the similarly-situated indices of face and zone multiplied, and all three added together, must be equal to 0:

$$u\,x+v\,y+w\,z=0$$

A numerical example makes the brevity still more apparent:

$$\begin{array}{lll} a\,b\,c\,.\,.\,.\,.\,.\,2\,1\,0 & & 2\ 1\ 0\ 2\ 1\ 0 \\ & & \times\times\times \\ p\,q\,r\,.\,.\,.\,.\,.\,1\,\bar{1}\,1 & & 1\ \bar{1}\ 1\ 1\ \bar{1}\ 1 \\ & & \hline \\ & & 1.1-0.\bar{1};\ 0.1-2.1;\ 2.\bar{1}-1.1 \\ & & 1-0; \qquad 0-2; \qquad -2-1 \\ u\,v\,w\,.\,.\,.\,.\,.\,[1\,\bar{2}\,\bar{3}] & & 1 \qquad\quad \bar{2} \qquad\quad \bar{3} \\ x\,y\,z\,.\,.\,.\,.\,.\,.\,3\,0\,1 & & 1.3+\bar{2}.0+\bar{3}.1=3-3=0 \end{array}$$

The face 3 0 1, therefore, lies in the zone $[1\ \bar{2}\ \bar{3}]$, produced by 2 1 0 and $1\ \bar{1}\ 1$.

Let us observe the method of zone-calculation according to Weiss:* Given two faces—

$$\boxed{a\,a:\beta\,b:n\,c} \quad \text{and} \quad \boxed{a'\,a:\beta'\,b:n\,c}$$

which are already reduced to a similar co-efficient of c. The zone produced is—

$$(n\,c;\ a''\,a+\beta''\,b)$$

therefore—

$$a''=\frac{a\,a'\,(\beta-\beta')}{a'\,\beta-a\,\beta'};\quad \beta''=\frac{\beta\,\beta'\,(a-a')}{a\,\beta'-a'\,\beta}$$

* Weiss, Berliner academische Abhandlungen, 1820–21, pp. 169–173.

The values $\alpha\,\alpha'$, $\beta\,\beta'$, are therefore negative when the axes a or b, for which they stand, are primed ($a'\,b'$).

If the face—

$$\boxed{a'''\,a : \beta'''\,b : n\,c}$$

lies in this zone, one of the following propositions must be right:

$$a''' : \beta'' = a'' \left\{ \begin{matrix} \beta''' - \beta'' \\ \beta'' - \beta''' \\ \beta''' + \beta'' \end{matrix} \right\} = \left\{ \begin{matrix} a''' - a'' \\ a'' - a''' \\ a''' + a'' \end{matrix} \right\} : \beta'''$$

The simple inspection of this method shows how minute in detail this method is. In the first place, the symbols of the faces, with respect to an axis, (in the above case c,) must be reduced to similar co-efficients; then by multiplication and addition, respective subtraction and division, the values a'' and β'' are to be determined. It is to be remarked that both the numerator and the denominator of these quantities are fractions, which must be reduced to a common denominator. The calculation, it is true, (*loc. cit.*, p. 169,) can be simplified when the symbols of the faces are written—

$$\boxed{\frac{1}{x}\,a : \frac{1}{y}\,b : n\,c}$$

This, however, is using Miller's symbols, which are the reciprocals of Weiss's; and even then the calculation is more circumstantial, because the three symbols are equated with reference to c, and are not symmetrical according to the three axes.

The steps of the calculation in the hexagonal system are still more incumbered, since, from a four-membered symbol a three-membered parameter must be first calculated, and then introduced into the previously-developed calculation.

Quenstedt* employs these symbols in his so-called zone-point formulæ in a somewhat more convenient, although in a much less concise, manner than Miller. Let there be three faces—

$$\boxed{m\,a : n\,b : c}, \boxed{p\,a : q\,b : c}, \text{ and } \boxed{x\,a : y\,b : c}$$

whose tautozonality is to be proved. For every pair of these the zone-point formulæ must be written, and the verification as to whether the zones are identical, made. Thus, for the zone—

$$\boxed{m\,a : n\,b : c} \text{ to } \boxed{p\,a : q\,b : c}$$

$$\frac{\frac{1}{q} - \frac{1}{n}}{\frac{1}{m\,q} - \frac{1}{p\,n}}\,a : \frac{\frac{1}{m} - \frac{1}{p}}{\frac{1}{m\,q} - \frac{1}{p\,n}}\,b$$

* Quenstedt, Mineralogie, 1863, p. 44.

The same must be done for the zone—

$$\boxed{ma : nb : c} \text{ to } \boxed{xa : yb : c}$$

$$\frac{\frac{1}{y}-\frac{1}{n}}{\frac{1}{my}-\frac{1}{xn}}\,a : \frac{\frac{1}{m}-\frac{1}{x}}{\frac{1}{my}-\frac{1}{xn}}\,b$$

from which, as a condition of tautozonality, follows the equality of both relations. Quenstedt and P. Klein* employ the zone-control in this form.

It is to be remarked that these zone-point formulæ can be essentially simplified, because the denominators of both sides are alike; thus—

$$\left(\frac{1}{q}-\frac{1}{n}\right)a : \left(\frac{1}{m}-\frac{1}{p}\right)b \text{ and } \left(\frac{1}{y}-\frac{1}{n}\right)a : \left(\frac{1}{m}-\frac{1}{x}\right)b$$

Also, the condition—

$$\left(\frac{1}{q}-\frac{1}{n}\right) : \left(\frac{1}{m}-\frac{1}{p}\right) = \left(\frac{1}{y}-\frac{1}{n}\right) : \left(\frac{1}{m}-\frac{1}{x}\right)$$

But this equation is much more complicated than Miller's. In our former example we had—

$$210 = \tfrac{1}{2}a : b : \infty c\,;\; 1\bar{1}1 = a : b' : c\,;\; 301 = \tfrac{1}{3}a : \infty b : c$$

Exchanging the axes a and c in all the three faces, in order to be able to make the co-efficient of c equal to unity, which has no influence on the tautogonality, we have—

$$\infty a : b : \tfrac{1}{2}c\,;\; a : b' : c\,;\; a : \infty b : \tfrac{1}{3}c$$

or—

$$\infty a : 2b : c\,;\; a : b' : c\,;\; 3a : \infty b : c$$

It follows that—

$$\frac{1}{m} = 0\,;\; \frac{1}{n} = \frac{1}{2}\,;\; \frac{1}{p} = 1\,;\; \frac{1}{q} = -1\,;\; \frac{1}{x} = \frac{1}{3}\,;\; \frac{1}{y} = 0$$

by substitution—

$$\left(-1-\frac{1}{2}\right) : \left(0-1\right) = \left(0-\frac{1}{2}\right) : \left(0-\frac{1}{3}\right)$$

or—

$$-\frac{3}{2} : -1 = -\frac{1}{2} : -\frac{1}{3}$$

The proportion is correct, consequently the zones exist. The numerical values of the letters must here, also, be substituted according to the above-mentioned method, and the division carried out; while in Miller's method the very simple and symmetrical calculation can be carried out on the indices, without the help of letters, by means of the crosswise multiplication and subtraction of whole numbers.

* Klein; Leonh. Jahrb., 1871, p. 480.

Naumann's method is still more circuitous: first, Weiss's parameters must be calculated, and then they must be introduced into the equation—

$$\frac{1}{a\,b'\,c''} + \frac{1}{b\,c'\,a''} + \frac{1}{c\,a'\,b''} = \frac{1}{a\,b''\,c'} + \frac{1}{b\,c''\,a'} + \frac{1}{c\,a''\,b'}$$

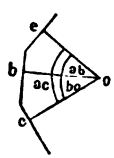
in which $a\,b\,c$, $a'\,b'\,c'$, $a''\,b''\,c''$, represent the parameters of the faces. If these numbers contain two figures, as is frequently the case in the hexagonal system, there must be twelve multiplications, six divisions, and the addition made. The division must be carried out to four decimal places, and sometimes farther; while in Miller's system the convenience of a calculation with whole numbers is always secured.

This circuitous course has caused the adherents of the schools of Naumann and Weiss, to this day, to use Quenstedt's method; and they are contented with an approximative zone-verification, while, since the foundation of Miller's method, even the beginner is both capable of and accustomed to verify every zone by means of the exceedingly simple calculation of zone-equations. In fact, Kohscharow,[1] in the year 1866, again first called attention to the zone-verification calculation, which, since the publication of Weiss, had been almost entirely forgotten; von Rath,[2] Hessenberg,[3] and C. Klein[4] followed, replacing the construction in specially-complicated cases by calculation.

The use of the angle of the normal to the faces, instead of the interior angle of the solid, is also important: in the first place, with respect to convenience and conciseness, while, as a rule, the interior angle is greater than 100°, and therefore contains three figures, the angles of the normals have, for the most part, two figures; further, the angles measured at present with the reflecting-goniometer are for the most part angles of normals. In the simple evaluation of an angle with the eye even the supplement is easier to estimate than the real angle, because it is generally smaller.

The most important advantage of normal angles is, that they can be immediately introduced into the calculation. This is especially apparent in tautozonal faces, in which, from two angles of every two out of three tautozonal faces, the third can be had by simple addition or substraction, (Fig. 1,) as—

Fig. 1.



$$\sphericalangle\, a\,b + \sphericalangle\, b\,c = \sphericalangle\, a\,c. \quad \sphericalangle\, a\,c - \sphericalangle\, a\,b = \sphericalangle\, b\,c$$

which is not the case with the angles made by the faces themselves.

In the determination of combinations a very quick orientation is furnished by this method. Lastly, only normal angles are suitable for introduction in spherical projections, where they themselves directly form the sides of the spherical triangle.

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[1] Von Kohscharow, Materialien zur Mineralogie Russlands, 1866, p. 216.
[2] Von Rath, Pogg. Ann., cxxxii, 1867, p. 398.
[3] Hessenberg, Min., Not. ix in Seuchenb. Ges., Abh. vii, 1870, p. 259.
[4] Klein, *loc. cit.*, p. 481.

This also shows one of the advantages of the method of spherical projections, which is entirely wanting in Quenstedt's system. Since, further, Miller's entire method of calculation is based upon spherical trigonometry, the illustrating figure is shown on the projection, which, therefore, at the same time represents the zone-connection of the form and the method of the calculation of the crystal.

Spherical projection has, finally, the great advantage of being limited, so that the geometrical situation of all faces can be actually delineated, and can be united in a comprehensive representation, a characteristic which is wanting both in the gnomonic method and that of Quenstedt. In this way alone it is possible to use projection for the introduction of all the physical relations, which circumstance, on account of its increasing use, is a very important one.

A reproach, which, although perhaps not expressed, still is silently made against this method of projection, is that in its construction triangles and dividers are necessary, while with Quenstedt's method triangles alone are used. This reproach is, however, entirely without foundation; for, in the first place, dividers are necessary for every exact projection, even if only the convenient form provided with steel points; but for general use both triangles and dividers are unnecessary, because on account of the extraordinary simplicity of zone-calculation the adherents of Miller's system of spherical projection are accustomed to use it only for representing and not for investigating existing zones, and they may therefore save themselves the trouble of making an exact drawing, unless they intend to publish.

To the many advantages of Miller's method no one has yet been able to oppose a disadvantage. If, in spite of this, it has not yet generally found its way into Germany and France, it is owing solely to the fact that in these countries Haüy, Weiss, and Naumann have taught; but when such completely independent theories are offered, the learner satisfies himself for the most part with the system which has been expounded; or if he afterward goes beyond that, the system to which he was first accustomed is easier, and his knowledge of it more fundamental, so that he does not become acquainted with many of the advantages of the new system.

The introduction of the Whewell-Miller principle was tried in Germany by Frankenheim, and in France by Bravais and de Senarmont, without, however, any permanent result. Recently the young german school, on account of the prominence to which the physical examination of crystals has attained, begins to make itself master of detached parts of Miller's method.

The purpose of the following pages is summarily to develop what is necessary for the solution of a combination, or for the knowledge of the physical nature of a crystal. We shall, therefore, in the first section, treat, according to Miller's method, the pure geometrical relations of a crystal, so far as they are requisite for the determination of combinations.

The second section treats of the possible systems of crystallization and their corresponding relations of symmetry; it is taken as an abstract from the work of von Lang. In the third section I have shown how, with the foundation of the optical relations of a crystal in general, the optical characters for each individual system of crystallization are derived from their symmetry.

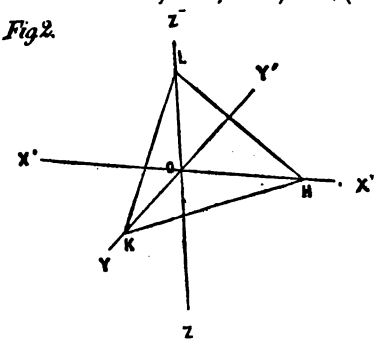
SECTION I.

THE GEOMETRICAL RELATIONS OF CRYSTALS.

§ 1.—Miller's Symbols.

It is well known that the situation of any plane is perfectly defined when its sections, o H, o K, o L, (Fig. 2,) of three straight lines, o X, o Y, o Z, which are not parallel, and which have a common origin, o, are known. These straight lines are called the axes; the point o the center of the axes; the plane of every two axes, X o Y, Y o Z, Z o Y, the planes of the axes; and the sections o H, o K, o L, the parameters of the face H K L.

Fig. 2.

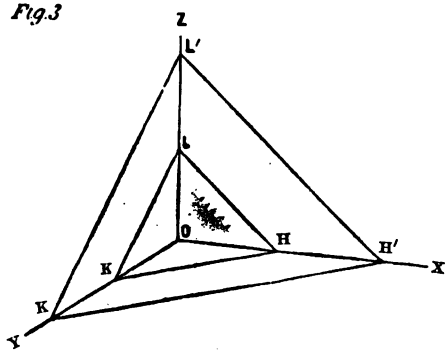


Because every axis considered in regard to O has two sides, these are distinguished as the positive and negative half-axes. For this reason the sections of the axes are used in the calculation as + o H or − o H.

The lines joining every two sections of the axes of a plane, (H K, K L, L H,) give the intersection of the plane H K L with the three planes of the axes.

If we multiply the three parameters of a face with the same number, the direction of the plane remains unchanged; it will only be moved parallel to itself, (Fig. 3.)

Fig. 3.



From the equality of the relation—

$$\frac{o\,H'}{o\,H} = \frac{o\,K'}{o\,K} = \frac{o\,L'}{o\,L} = m$$

results the similarity of the triangles K O L, K′ O L′, &c., and from this the parallelism of H K L and H′ K′ L′.

If another face, A B C, is given, with the parameters o A, o B, o C, which we may call a, b, c, then—

$$o\,A = a;\ o\,B = b;\ o\,C = c$$

and the face H K L is determined when the relations—

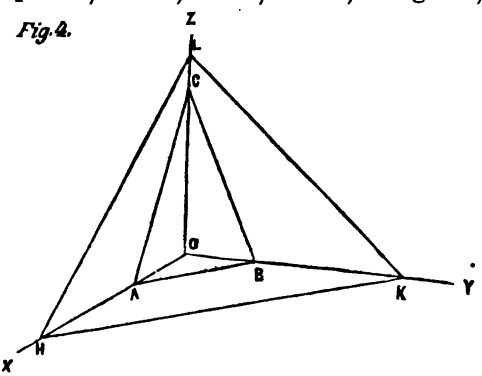
$$h = \frac{o\,A}{o\,H} = \frac{a}{o\,H};\quad k = \frac{o\,B}{o\,K} = \frac{b}{o\,K};\quad l = \frac{o\,C}{o\,L} = \frac{c}{o\,L}$$

are known; so a third face, H′ K′ L′, is determined by its relations or indices h' k' l', in which—

$$h = \frac{a}{o\,H'};\quad k' = \frac{b}{o\,K'};\quad l' = \frac{c}{o\,L'}$$

We see, also, that if three planes, X o Y, Y o Z, Z o X, are given, whose three lines of section represent the axes o X, o Y, o Z; further, a fourth face, A B C, whose section of these axes is the measure of the length of the axes, any face in their direction is perfectly determined when its indices, *i. e.*, the relation between the parameters of A B C and its own, are given.

Fig. 4.



The values a b c and the plane of the axes are constant for one and the same crystal.

Respecting the indices h, k, l, certain important cases are to be distinguished:

I. All three of the indices may be different from (o h, k, l,) $\gtrless$ o. This is the general case, and represents octahedral or pyramidal faces.

II. One index, l, for instance, equals zero, $l = o$; the face h, k, o, is evidently parallel to the axis o Z, and we have—

$$l = \frac{o\,C}{o\,L} = \frac{c}{o\,L} = o$$

Because o C $= c$ is constant, this fraction can only be equal to o if o L is infinitely great; but if the face H K o cuts the axis o Z at an infinite distance, it is parallel to it. Thus, if $k = o$, we have h o l, and if $h = o$, we have o k l of the axis of Y, parallel faces with respect to X. These kinds of faces are called dodecohedral, prismatic, or dome-faces.

III. Two indices $= o$ $k = l = o$....1 0 0; $l = h = o = 0$ 1 0; $h = k = o$ ....0 0 1, the face 1 0 0 has first the index $k = o$, and is for that reason parallel to the above axis of Y, and also to the axis of Z, because $l = o$. This face contains, therefore, both the axes of Y and Z. It is with them parallel to the axis-plane X o Z. We call such faces pinacoids; they are those by means of whose section-line the position of the axes is determined.

If the planes of the axes are parallel faces, X *o* Y, Y *o* Z, Z *o* X, as the faces A B C and H K L, which may be real or possible faces of a crystal, experience shows that the indices *h k l* of every possible face of this crystal are to each other as rational numbers.

This law, which is the first fundamental law of crystallography, is called the law of rational indices; it is of the greatest importance, and allows of the derivation of the greater part of the other laws of crystallography.

If the indices *h k l* of any face of a crystal are rational, it is always possible to represent them by three positive or negative whole numbers, because the direction of a plane remains unchanged when its three indices are multiplied by the same number.

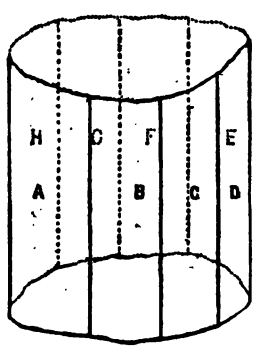
Experience shows further that the indices of the most frequently occurring faces are almost always the simplest whole numbers 0 and 1, rarely 2, so that the calculation with them will always be very simple.

§ 2.—Law of Zones.

The consideration of the zones occurring in a crystal is of the greatest importance for the determination of a combination.

Two planes which are not parallel always cut each other, when duly extended, in a straight line; all planes, therefore, whose lines of section are parallel to the same straight line, belong to a zone, and are called tautozonal faces; the straight line to which their lines of section are parallel is called the axis of the zone. (Fig. 5.)

Fig. 5



Because the axis of a zone is parallel to all the faces of that zone, a plane, P, perpendicular to the axis of the zone, will also be perpendicular to all the faces of that zone; and when a perpendicular to every zone-face is erected, all of these normals will be parallel to this face P. This important characteristic of tautozonal faces, that their normals all lie in a plane perpendicular to the zone-axis, we shall make use of in the discussion of spherical projection.

After the direction of the zone-axis is determined by the section of two planes which are not parallel, it must be possible, from the known elements of these planes, to calculate for the indices such values as will be characteristic for the axes of the zone produced by these planes. Let P (*h k l*) and Q (*p q r*) be the two planes, and let their indices be written twice, one over the other, and multiplied crosswise, beginning with the second upper index *k*—

$$\begin{array}{c} h\ k\ l\ h\ k\ l \\ \times\ \times\ \times \\ p\ q\ r\ p\ q\ r \\ \hline kr - lq;\ lp - hr;\ hq - kp \\ u \qquad v \qquad w \end{array}$$

Subtracting now the products obtained by multiplying the index right above with that left below, from that obtained by multiplying the index left above with that right below, we obtain three whole numbers $(u\,v\,w)$, which are either positive or negative, are determined for the zone P Q, and are called zone-indices. In order to distinguish these from the indices of the faces, they are inclosed in rectangular brackets.

The zone-indices of a zone containing more than two faces can be calculated from any two faces of the zone which are not parallel. The same value is always obtained, abstraction being made of a constant factor of all three indices, with which we can always multiply all of them without changing the direction of the face or line represented.

If, now, a third face, R(xyz), is placed in the above zone P O, we have a simple criterion, whose expression is produced from the fact that the zone-axis [P R] or [Q R] must have the same indices, even to a constant factor, as [P Q]. This criterion is the existence of the equation—

$$ux + vy + wz = o$$

If this equation is realized, all the three faces P Q R are in the same zone. If the symbols of two zones, $[efg]$ and $[u\,v\,w]$, are given, the symbol of a face (xyz) lying in both zones may again be found by crosswise multiplication—

$$\begin{array}{c} e\ f\ g\quad e\ f\ g \\ \times\ \times\ \times \\ u\ v\ w\quad u\ v\ w \\ \hline fw - gv;\ gu - ew;\ ev - fu \end{array}$$

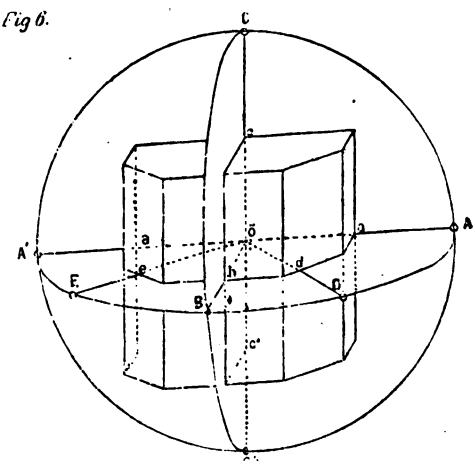
in the same way as the zone-symbol from the indices of two faces.

At the close of this section the most important special zone-laws and some examples of the development of zones will be given.

§ 3.—Spherical Projection.

The method of spherical projection introduced by Naumann gives the simplest means of representing the opposite faces of a crystal. It has the advantage of showing, even in extremely rough executions of it, a representation of the zone-combinations of a crystal, and allows of the determination of the indices of its faces, on the assumption of a primitive form, almost without any measurements.

Fig 6.



For this purpose let us imagine that from a point o, in the interior of a crystal, (Fig. 6,) perpendicular straight lines, $o\,a$, $o\,a'$, $o\,b$, $o\,c$, $o\,c'$, $o\,d$, $o\,e$, be drawn to all of its faces. From the point o, as a center, let us construct a sphere of any radius, and

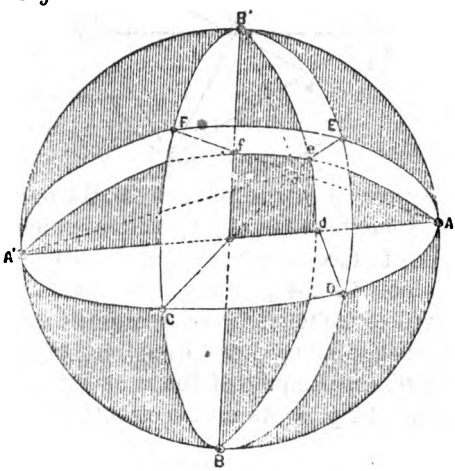
produce the perpendiculars until they cut the sphere A A′ B C C′ D E, &c., which are called the poles of the faces, which they meet.

In this construction, in which, for the sake of distinctness, only the front side is drawn, we see immediately that the poles of tautozonal faces, A D B E A′ for instance, lie in a great circle of the sphere, because the normals of tautozonal faces lie in a plane, which must pass through *o*, from which point all the normals are produced; a plane passing through the middle point, however, cuts the sphere in a great circle, which consequently contains the pole of the tautozonal faces.

In order to draw a sphere containing the poles of the faces of a crystal, we may select several different methods of projection. Of these the stereographic method, introduced by Miller, is the most convenient.

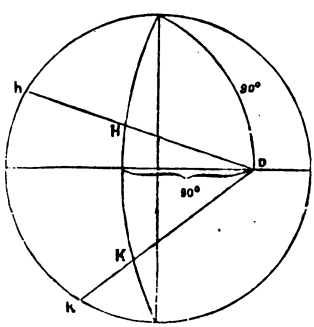
As plane of projection let us take, for this purpose, a plane passing through the center of the sphere *c*, (Fig. 7) which, according to the above, cuts the sphere in a great circle, A B C; let us draw a diameter of the sphere, O C, perpendicular to this, whose extremities, O and C, are 90° from every point of the principal circle, so that the lower pole O shall be the point of sight; let us now join by a straight line every pole of the sphere A B C D E F .... with the point of sight O. The intersections A B *c d e f* .... of these straight lines with the principal circle give the stereographic projection of the pole A B C D.

Fig 7.



In general, the principal circle will be taken perpendicular to the faces of a zone, so that the projection of these points of the faces will be the periphery of the circle.

Fig 8.



The most important peculiarities of such a projection are the following:

1. Every circle will be projected on the sphere, either as a circle or a diameter.

2. Every great circle will be projected on the sphere as an arc, which cuts the principal circle in the extremities of a diameter of the zone, or as a diameter itself. In such an arc, for that reason, also, the poles of the tautozonal faces lie, as, for instance, A *e f* A′; B *d e* B′; B *c f* B′; A *d c* A′.

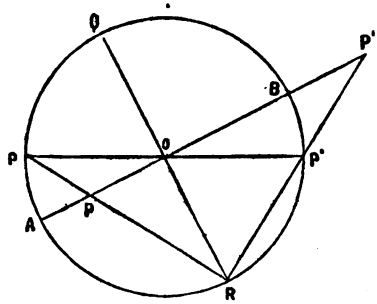
3. Let every point, P, which, on the sphere, is at 90° from all points of this circle, be the pole of a zone-circle, H K, (Fig. 8,) which is also the

projection of a face perpendicular to the zone-faces. The proposition obtains that the normal angle of two faces, H and K, is equal to the arc *h k*, which is cut off from the principal circle by the straight lines P H and P K produced.

From these three characteristics all the laws for the construction of stereographic projection are derived.

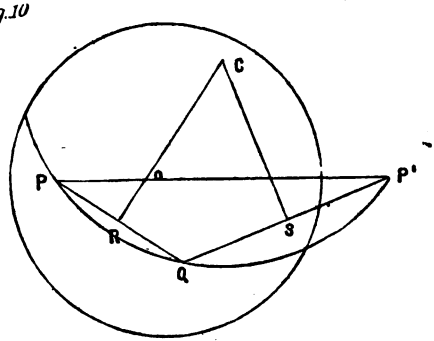
It is immediately apparent that the normal angle of all faces projected by points on the principal circle are determined by the arcs contained between the poles; that all zones passing the center of the main circle will be projected as diameters; that, further, the pole of such a zone falls again in the principal circle, and will be on one of the extremities of the diameter perpendicular to the zone.

Fig. 9.



If the projection of a pole, P, (Fig. 9,) is given, and that of the opposite face parallel to it sought, it is at once clear that it must lie outside of the principal circle. If a zone is determined by P and the center, *o*, of the main circle, the opposite pole P′ must be in the same zone, because every zone in which a face lies must also contain the opposite face which is parallel to it. In the zone P O we have now only to look for the point at 180° from P in order to determine P′. For this purpose we must, according to the third characteristic of projection mentioned above, draw from one of the points R or Q, which, as before, represent the pole of the zone P O, the point R for instance, a straight line, R P *p*, to its intersection with the principal circle; find the point *p′* of the principal circle, which is, at the required angle, 180°, from *p*; and then draw a straight line, R *p′* P′, whose section with the zone P O gives the pole opposite to P.

Fig. 10

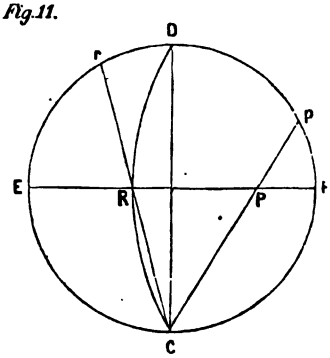


If two poles, P Q, (Fig. 10,) be given, and the zone passing through them be sought, we look for the opposite pole of one of them, P′ for instance, which in any case must lie in the zone P Q. Through the three points P Q P′ we draw, according to the known method, (erection of a perpendicular in the middle of a line joining any two points,) an arc, which represents the required zone.

In order to find the pole of a given zone, C R, (Fig. 11,) we must consider that it must be 90° distant from every point of the zone-circle. If,

now, C,D are the points of section of the zone with the main circle, we draw the diameter C D and a perpendicular to it, E F, and it is clear that the pole sought for must lie in the zone E F. Since, now, it must be 90° distant from every point of the zone, and therefore also from R, while the pole of the zone E F is one of the points C or D, we draw the straight line C R r and C P p, so that the arc $r\,p = 90°$, and thus find the pole p of the zone C R D.

Fig. 11.



Thus, all the expedients are given which are necessary for the construction and use of the projection; in general, the simplest of these are sufficient, especially while in this method of projection we do not aim at the greatest exactitude attainable, but only a presentable representation of the arrangement of the faces.

As a close of this section we shall give some special modes of the laws of zones, and an example of a complete development of them.

1. Zone passing through two pinacoids—

$$\begin{array}{l} 1\,0\,0 \quad 1\,0\,0 \\ 0\,1\,0 \quad 0\,1\,0 \\ \hline 0.0-0.1\,;\; 0.1-1.0\,;\; 1.1-0.0 \\ \quad\; 0 \qquad\quad 0 \qquad\quad 1 \end{array}$$

[0 0 1] is the symbol of the third pinacoid. If a face, $h\ k\ l$, lies in this zone, so must—

$$h.o + k.o + l.1 = o$$

also, $l = 1$, the general symbol of a face lying in the zone $1\,0\,0 . 0\,1\,0 =$ [0 0 1] is $h\,k\,o$.

2. Zone passing through a pinacoid and any face:

$$\begin{array}{l} h\,k\,l \quad h\,k\,l \\ 1\,0\,0 \quad 1\,0\,0 \\ \hline k.o-l.o\,;\; l.1-h.o\,;\; h.o-k.1 \\ \quad\; o \qquad\quad l \qquad\quad \bar{k} \end{array}$$

If a third face, $x\,y\,z$, lies in the zone $[o\ l\ \bar{k}]$, so must—

$$x.o + y.l - k.z = o$$

or—

$$y\,l = kz\,;\ \frac{y}{z} = \frac{k}{l}$$

If, therefore, a zone passes through a pinacoid, the relation of those two indices, which, in the symbol of the pinacoid, are o, is constant for all the faces of this zone.

3. The cases given under the second and third rules are special cases of a more general one; and, certainly, two given faces, $(h\,k\,l)$ and $(p\,q\,r)$, in which—

$$\frac{k}{l} = \frac{q}{r}$$

can always be so represented that their symbols have the form $(e\,u\,v)$ and $(x\,u\,v)$, because the three indices of a face may be multiplied by the same number without changing the symbol.

For the zone we have—

$$\begin{array}{l} e\,u\,v \quad e\,u\,v \\ \underline{x\,u\,v \quad x\,u\,v} \\ u\,.\,v - v\,.\,u\,;\; v\,.\,x - e\,.\,v\,;\; e\,.\,u - u\,.\,x \\ \qquad o\,; \qquad v\,(x-e)\,;\; u\,(e-x) \end{array}$$

or, if we divide the three zone-indices by $(x-e)$, $[0\,v\,\bar{u}]$; a face, $(r\,s\,t)$, lies in this zone, if—

$$o\,.\,r + v\,.\,s - u\,.\,t = o$$

so—

$$\frac{s}{t} = \frac{u}{v}$$

Let any two faces of a zone be represented by the symbols $(x\,u\,v)$ and $(e\,u\,v)$, or, generally, let them have two similarly-situated indices in both faces with like relations, all the faces of this zone will be represented in the form $(p\,u\,v)$.

That the second law comes also under this head is clear, because the relation $\frac{o}{o}$ is indeterminate, and therefore can answer to every value.

As an example of development by zones, we have chosen the crystal represented in Fig. 12. Because we assume that there are no measure-

Fig. 12.

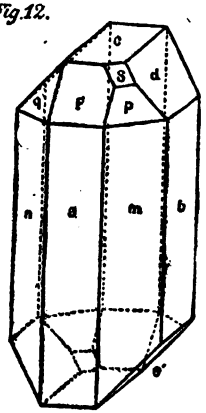
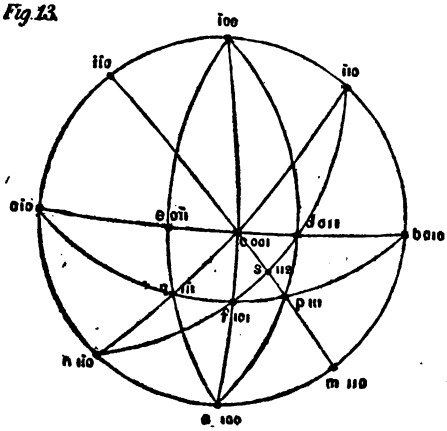


Fig. 13.



ments, but only the data of the zones, we shall presume, in the projection, (Fig. 13,) that it is triclinic. In this projection we record the faces in the order in which they are to be determined.

Let the zones to be determined be—

b m a n; b d c e; a f c; a p d; b p f q; c s p m; d s f n; c q n; a q e

The existence of these will be seen principally from the parallelism of the respective edges. Where there is no real edge, as is the case in the angle *a q*, the hypothetical zone-axis can be found by turning the crystal round. All faces which, in turning round the same axis, reflect the light are tautozonal.

In order to determine the combination, it is first necessary to select a system of axes. Regard will be had to the real or apparent symmetry of the crystal, in this way, that when a system less symmetrical in completeness and inclination of the faces approaches one of higher symmetry, this analogy is retained.

We select three faces, *a b c*, for the planes of the axes; their lines of section give the crystallographic axes. We project these in such a way that the zone *a b* is contained in the principal circle.

The exactitude of the relation of the angles makes naturally no difference, if it is only a question of the solution of the combination. The faces are introduced into the projection in the order in which they are to be determined, first *a b c*.

The faces *a b c* then are designated by the symbol belonging to the pinacoids, 1 0 0, 0 1 0, 0 0 1.

In order to fix a ground-form, we have yet to determine the relations of the axes; this, according to the relation of *p*, may be (1 1 1); the axes-sections of the face *p* give also the value *o* A, *o* B, *o* C, from which the parameters of every other face will be determined.

That the indices of *p* must be 1 1 1 follows from the equation (p. 9) in which the indices of a face are determined, as—

$$h = \frac{o\,\mathrm{A}}{o\,\mathrm{H}};\quad k = \frac{o\,\mathrm{B}}{o\,\mathrm{K}};\quad l = \frac{o\,\mathrm{C}}{o\,\mathrm{L}}$$

Substituting the section *o* A, *o* B, *o* C, in this equation, we have—

$$h = k = l = 1$$

After the outline and the axes of the crystal are determined, the drawing of the faces can be developed.

Determination of *m*. *m* lies in the zones *b m a n* and *c . . . . p m*. In order that a face may lie in the first zone, it is a necessary and sufficient condition that it has the symbol *h k o*, that is, is parallel to the axis *c*, as also follows from the derivation of the zone-equation.

For the second zone we have the condition—

$$\frac{k}{h} = 1$$

because, as we have seen, the equality of the same index-relations, in two faces of a zone, determines their equality for all the faces of that zone; thus—

$$\frac{h}{k} = \frac{1}{1} = \frac{0}{0}.$$

This results also from the zone-equation—

$$\begin{array}{l} 1\,1\,1 \quad 1\,1\,1 \\ 0\,0\,1 \quad 0\,0\,1 \\ \hline 1\,.\,1-0\,.\,1;\ 1\,.\,0-1\,.\,1;\ 1\,.\,0-0\,.\,1 \end{array}$$

which gives $[1\,\bar{1}\,0]$ as the zone-equation, or—

$$1\,.\,x-1\,.\,y+0\,.\,z=0 \text{ or } x=y$$

as condition of the tautozonality of a face, $x\,y\,z$, with 0 0 1 and 1 1 1; the symbol of $m\,h\,k\,o$ becomes changed under these circumstances into (1 1 0).

In the same way the position of d, in the zones $b\,d\,c$ and $a\,p\,d$, is determined. The first zone gives, as condition, the first index as equal to 0, it is thus $o\,h\,l$; the second gives the equation of the second and third index—

$$\frac{k}{l}=\frac{1}{\bar{1}}=\frac{0}{0}=1$$

and therefore the symbol (0 1 1).

Finally, the face f is determined in the same way by the zone $a\,f\,c$, as $h\,o\,l$, and by the zone $b\,p\,f$, as 1 0 1, because—

$$\frac{h}{l}=\frac{1}{\bar{1}}=\frac{0}{0}=1$$

Thus it is to be kept in view that the quotient $\frac{0}{0}$ may have any rational value which is first fixed by the two faces.

For the face n we have the zone $b\,m\,a\,n$, by which we get the symbol $k\,h\,o$ and $d\,f\,n$; for the last we have—

$$\begin{array}{l} 0\,1\,1 \quad 0\,1\,1 \\ 1\,0\,1 \quad 1\,0\,1 \\ \hline 1\,.\,1-0\,.\,1;\ 1\,.\,1-0\,.\,1;\ 0\,.\,0-1\,.\,1 \end{array}$$

or $[1\,1\,\bar{1}]$; also as condition—

$$h\,.\,1+k\,.\,1-0\,.\,1=0 \text{ or } h=-k$$

This condition is satisfied by $1\,\bar{1}\,0$ and $\bar{1}\,1\,0$, of which the first is the symbol for the face in front, and the last for the opposite one behind.

For the determination of q we have the zones $c\,q\,n$ and $d\,p\,f\,q$; the first gives, when $h\,k\,l$ is the symbol of q—

$$\frac{h}{k}=\frac{1}{-1}=\frac{0}{0}=-1 \text{ or } (h\,\bar{h}\,l)$$

the last—

$$\frac{h}{l}=\frac{1}{\bar{1}}=\frac{1}{\bar{1}}=1 \text{ or } (h\,\bar{h}\,h)$$

which, when contracted, is $1\,\bar{1}\,1$.

The face e lies in the zone $b\,d\,c\,e$, wherefore $h=o$; and in $a\,q\,e$, for which reason—

$$\frac{k}{l}=\frac{-1}{1}=\frac{0}{0}=-1$$

e has thus the symbol $(0\,\bar{1}\,1)$.

There remains s in the zones $m\,p\,s\,e$ and $d\,s\,f\,n$ to be determined; the first zone gives—

$$\frac{h}{k}=\frac{1}{\bar{1}}=\frac{1}{\bar{1}}$$

or the general symbol $h\,h\,l$; and has $[1\,1\,\bar{1}]$ for its zone-index, so—

$$h + h - l = 0 \text{ or } 2\,h = l$$

which condition is satisfied by (1 1 2).

Thus the collective forms of this combination are determined.

There certainly may be cases presented where the existing zones do not suffice to determine all the faces of a combination; these cases are, however, rare, and occur in very few instances.

Instead of the above selection of a face, (1 1 1), determining the collective relations of the axes, two domes in two pinacoid zones could very well be used, as 1 1 0, in which $a : b$, and 1 0 1, by which $a : c$, is determined.

In the simpler and often recurring faces, as we have seen above, even the very simple calculation of the symbols from two zone-symbols, by crosswise multiplication, is superfluous, because at least the conditions for it, in a zone, can be at once expressed in the general symbol of the face, so that by substitution in the equation—

$$hx + ky + lz = 0$$

the indices $h\,k\,l$ are fully determined.

SECTION II.

SYMMETRY OF THE SYSTEMS OF CRYSTALLIZATION.

§ 1.—DERIVATION OF THE SYSTEM FROM THE LAW OF RATIONAL INDICES.

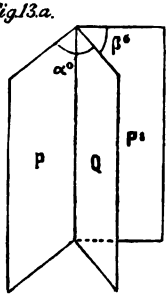
The rationality of the indices is, for the possibility of a face of a crystal, as we have said above, not only a necessary but a sufficient condition. It is, therefore, a possibility of every face whose indices are rational numbers. A collection of faces, therefore, which is to obey the law of rational indices must also answer to all the consequences which in mathematics follow from this law.

The carrying-out of this deduction, which can here only be announced, leads us to the different elements of symmetry, and especially to the consideration of planes of symmetry.

A plane of symmetry has the peculiarity that its physical relations are equal on both sides of it.

The identity of the physical peculiarities of two faces or lines is also determined by the similarity of their position with respect to the plane of symmetry, and this condition is really fulfilled by two planes when they are tautogonal with the plane of symmetry, and are so situated with regard to both sides that they form like anzles with them, (Fig. 13*a*, where the angle $P : Q = \alpha^\circ$ and $P^1 : Q = \beta^\circ$ are equal to each other.) Two lines, *o* A and *o* B, (Fig. 13*b*,) satisfy the condition if they, with respect to the plane of symmetry P, contain a similar angle; and if a plane, R, at right angles to the plane of symmetry can be passed through, then arc A C = arc C B.

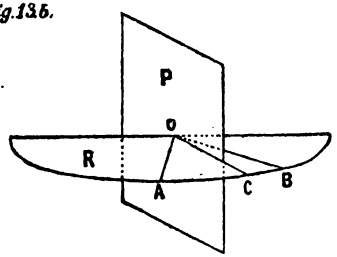
Fig. 13.a.



The derivation of the crystalline system is as follows: Let two possible faces of a crystal be taken which are symmetrical with respect to a

plane; it is to be determined if symmetry with regard to another plane does not follow from it; given a zone, which is symmetrical with respect to one or more faces, *i. e.*, that for every face of the zone there is also a possible one which will be symmetrical with it, it is required if from this, symmetry in other directions does not follow, *i. e.*, if for every possible face of the given zone another face is not also possible, with which, according to presumption, the zone sought for is parallel.

Fig. 13 b.



The criterion of the possibility of a face is, therefore, always the rationality of its indices. Proceeding in this way, we recognize that only that reunion of faces is, crystallographically speaking, possible, which, by the number and position of their planes of symmetry, belong to one of the seven characteristic crystalline systems. By plane of symmetry of a crystal we understand a plane in relation to which all the possible faces of a crystal are symmetrical, so that for every possible face of a crystal there is another which, so far as the plane of symmetry is concerned, is symmetrical with it. It is therefore apparent, that only the following combinations are possible:

Fig. 14.

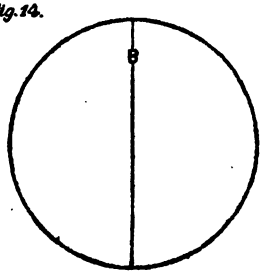
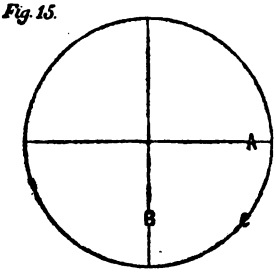


Fig. 15.



1. No plane of symmetry existing..... TRICLINIC SYSTEM.
2. One plane of symmetry, B, (Fig. 14)..... MONOCLINIC SYSTEM.
3. Three different planes of symmetry at right angles to each other, A, B, C, (Fig. 15)..... RHOMBIC (ORTHORHOMBIC) SYSTEM.

Fig. 16

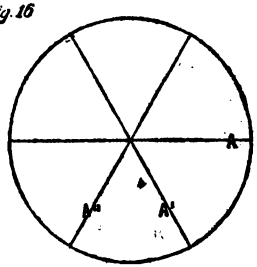
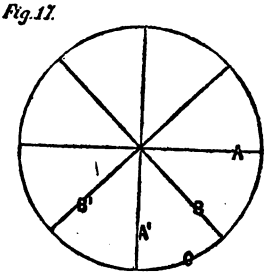


Fig. 17.



4. Three tautozonal and similar planes of symmetry inclined to each at angles of 60° and 120°, A A′ A″, (Fig. 16), RHOMBOHEDRAL SYSTEM.

5. Five planes of symmetry, of which four are tautozonal and inclined to each other 45° and 90°, every two 90° apart similar, A A′, A′ A″, (Fig. 17.) The fifth, C, at right angles to all the others, and not similar..... TETRAGONAL SYSTEM.

6. Seven planes of symmetry, six of which are tautozonal and inclined 30° and 60°, every three 60° apart, similar, A A′ A″, B B′ B″, (Fig. 18.) The seventh, C, at right angles to all the others, not similar..... HEXAGONAL SYSTEM.

Fig. 18.

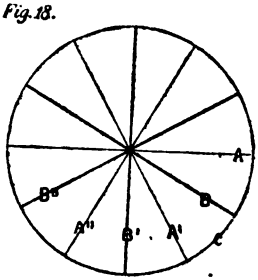
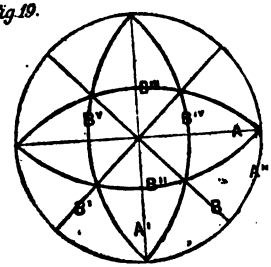


Fig. 19.



7. Nine planes of symmetry, three of which, A A′ A″, (Fig. 19,) are at right angles to each other, and similar. The other six, similar to each other, B B′ B″ .... B^{v}, intercalated between every two tautozonal A, and at an angle of 45°. TESSERAL (ISOMETRIC) SYSTEM.

§ 2.—CHARACTERISTICS OF THE SYSTEMS.

From the above statement of the relations of symmetry in each crystalline system we shall next derive the single faces belonging to each form as well as the most practical method of selecting the axes of the crystal.

For axes we may select any three edges or zone-axes which are formed by three possible faces of the crystal not tautozonal with each other.

We shall, however, on account of the existence of planes of symmetry, so select the axes that, wherever it is possible, they are placed symmetrically to the planes of symmetry, by which we shall at once see that all the faces of a form will take the same numerical indices, but arranged in different orders. We understand by form the combination of all those faces which are symmetrical with each other, according to the planes of symmetry of the given crystal, and which, together, possess the same physical peculiarities.

With regard to the selection of the axes, we only remark that it appears necessary, on theoretical grounds, which were first developed by Frankenstein, so to select the axes that every acute axis-angle shall be greater than 60°, and that every obtuse one shall be less than 120°, which is always possible.

1. TRICLINIC SYSTEM.—No plane of symmetry. The choice of the axes is arbitrary, as also the face 1 1 1, by which the plane of the axes is determined—

$$a \gtrless b \gtrless c\,;\ \xi \gtrless \eta \gtrless \zeta$$

Five elements are undetermined, two relations and three angles of the axes. Because no plane of symmetry exists, a single face, $h\,k\,l$, (Fig. 20,) with the one parallel to it, constitutes a form. It is only necessary to consider this analogy in the selection of the axes, where there exists a

Fig. 20

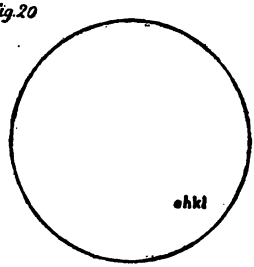
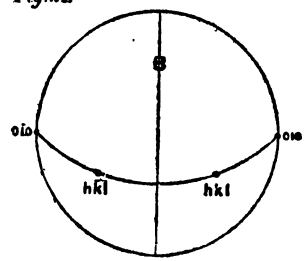


Fig. 21.



similarity in the angle and in the composition of the faces, with a more highly symmetrical system, as the monoclinic or orthorhombic.

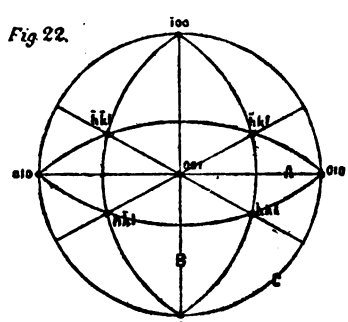
2. MONOCLINIC SYSTEM.—One plane of symmetry, B, (Fig. 21.) We first select this plane as one of the planes of the axes, especially for the plane X Z; so that it takes the symbol 0 1 0. For every face, $h\,k\,l$, a second one is now possible, which, with it, is placed symmetrically with regard to the plane of symmetry 0 1 0, and, therefore, as is easily seen, takes the symbol $h\,k\,l$. These two faces, with those opposite to them, constitute together the general form of the monoclinic system. A zone-axis is determined by every two such pairs of faces, which, as is easily perceived, must lie in the plane of symmetry, because 0 1 0 lies in the zone $[(h\,k\,l)\ (h\,\bar{k}\,l)]$. If two such zone-axes are taken for the axis of X Z, it is at once clear that the angles of the axes will be—

$$X\,Y = \zeta = 90^\circ;\ Y\,Z = \xi = 90^\circ;\ (X\,Z = \eta) \gtrless 90^\circ$$

A fourth face gives the sections of the axes $a \gtrless b \gtrless c$, and we have in this system three unknown elements, two ratios, and one angle of the axes.

3. ORTHORHOMBIC SYSTEM.—Three planes of symmetry, A B C, (Fig. 22,) at right angles to one another, which we select for the planes of the axes, with the symbols 1 0 0, 0 1 0, 0 0 1. The three axes will, for this reason, be at right angles to each other, and we have now, by means of a fourth plane, to determine their lengths, so that—

Fig. 22.

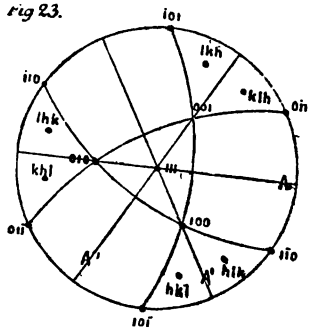


$$a \gtrless b \gtrless c;\ \xi = \eta = \zeta = 90^\circ$$

In this system we have, therefore, two unknown elements, $\frac{a}{c}$, $\frac{b}{c}$; the four faces, $h\,k\,l$, $\bar{h}\,k\,l$, $h\,\bar{k}\,l$, $\bar{h}\,\bar{k}\,l$, with their opposites, are similar, so that the general form is an eight-sided rhombic pyramid.

With regard to the selection of the three pinacoids, there may be a number of assumptions. Grailich and Lang take $a > b > c$; Schrauf selects, in substances which can be optically examined, 0 0 1, perpendicular to the bisectrix, 1 0 0 and 0 1 0, so that $a > b$; other authors follow no principle, but take the first method of exhibition.

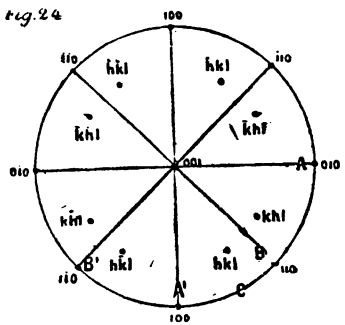
4. RHOMBOHEDRAL SYSTEM.—Three planes of symmetry, A A′ A″, (Fig. 23,) which are tautozonal, similar, and inclined to each other at an angle of 60°. In this case it is not admissible to select the planes of symmetry for the planes of the axes, because they are tautozonal. In order to observe the symmetry of the method of notation, we select for the planes of the axes three faces of the crystal which are symmetrically situated with regard to the planes of symmetry, and so constitute a form. The faces 1 0 0, 0 1 0, 0 0 1, must be perpendicular to every plane of symmetry, because only one such form, composed of only three faces with their opposites, exists; every other one is composed of six or of two. For the determination of the planes of the axes we select a face, as 1 1 1, which is at right angles to the zone-axis of the planes of symmetry, and is consequently similarly inclined to the three planes of the axes. Therefore—



$$a = b = c; \quad (\xi = \eta = \zeta) \gtrless 90°$$

A single dimension, the angle of the axes, is undetermined.

The three planes of symmetry have the symbols $1\,0\,\bar{1} = A$; $0\,1\,\bar{1} = A'$; $\bar{1}\,1\,0 = A''$. The symbol of each, with the faces tautozonal to the plane of symmetry, which are prisms according to the general notation, thus deviating from usage in the other crystalline systems, is liable to the condition $h + k + l = o$, because the symbol of the zone of symmetry is [1 1 1]. The other forms are scalenohedrons, which is the general form of this system, with six faces, $h\,k\,l$, (Fig. 23,) and their opposites; rhombohedrons, whose faces are perpendicular to every plane of symmetry; the base 1 1 1.



It is plain that the axis-angle ξ is equal to plane-angle of the faces at the vertex of the primitive rhombohedron, (1 0 0).

5. TETRAGONAL SYSTEM.—Four tautozonal planes of symmetry inclined at an angle of 45° to each other; every alternate two, A A′, B B′, (Fig. 24,) similar; a fifth one, C, perpendicular to these, but not similar. For planes of the axes we select two similar planes of symmetry, which are perpendicular to each other, B A A′, and the single plane of symmetry, C, at

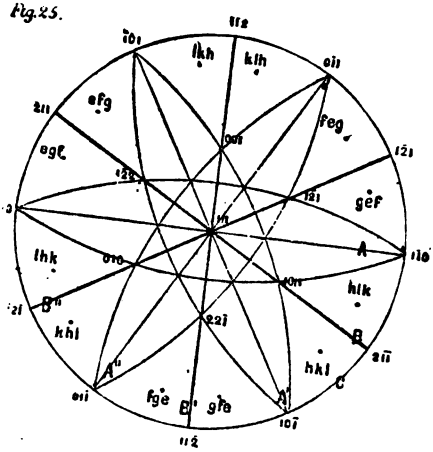
right angles to it, and finally 0 0 1 as plane of X Y. For the determination of the lengths of the axes we select a face, 1 1 1, perpendicular to one of the intermediate planes of symmetry. We thus have the elements—

$$\xi = \eta = \zeta = 90°;\ (a = b) \gtrless c$$

Thus we have only one unknown quantity, $\frac{a}{c}$. The intermediate planes of symmetry have the symbols 1 1 0, $1\bar{1}0$. The most general form is a pyramid of sixteen faces. The similar faces of $h\,k\,l$ may be seen in Fig. 24.

6. HEXAGONAL SYSTEM.—Seven planes of symmetry, six of which are tautozonal and inclined at an angle of 30°; every other one, A A′ A″, B B′ B″, (Fig. 25,) similar; and the seventh, which is at right angles to them, not similar. We might here have selected for the planes of the axes three planes of symmetry, as C, and two others from the zone, symmetrical to the planes of the axes, but the symmetry of the notation would thus be lost. We select, therefore, as in the rhombohedric system, three alternate faces, of a form perpendicular to the six planes of symmetry, for the planes of the axes 1 0 0, 0 1 0, 0 0 1. We determine the length of the axes, as in the rhombohedric system, by the face 1 1 1, which is perpendicular to the axis of the zone of symmetry, by means of which we get, as before—

Fig. 25.



$$a = b = c;\ (\xi = \eta = \zeta) \gtrless 90°$$

Because in particular values of their elements there is no difference between this and the rhombohedric system, they are often united, which is contrary, however, to physical laws.

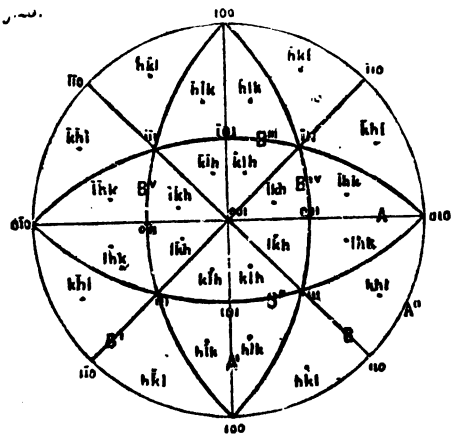
In this system it is no longer possible to represent the united faces of a form with the same indices with regard to the symbols of the planes of symmetry, as $1\bar{0}\,\bar{1}$, $0\,1\,\bar{1}$, $\bar{1}\,1\,0$, it is for the primary $1\,1\,\bar{2}$, $\bar{1}\,2\,\bar{1}$, $\bar{2}\,1\,1$; for the secondary planes, B B′ B″, whose sign follows from the zones, we have, for the faces $e\,f\,g$, belonging to those lying opposite to $h\,k\,l$, the determinative equations—

$$\begin{aligned} e &= -h + 2k + 2l \\ f &= 2h - k + 2l \\ g &= 2h + 2k - l \end{aligned}$$

The most general form of this system is a twenty-four-faced pyramid, the half of whose faces, as is seen on Fig. 25, are represented by the

symbols $h\,k\,l$, and the other half by $e\,f\,g$. The forms of this system are generally pyramids of twenty-four faces, two orders of twelve-faced pyramids, whose faces are perpendicular to the principal section, prisms of twelve faces, two orders of six-faced prisms, and the base.

7. TESSERAL (ISOMETRIC) SYSTEM.—Nine planes of symmetry, three of which, A A′ A″, (Fig. 26,) are similar and at right angles to each other; the others tautozonal in pairs, with an A intercalated between each two, B . . . . B^{v}, at an angle of 45° to them. We select the three which are perpendicular to each other for the plane of the axes, and determine the length of the axes by the face 1 1 1, which lies in an intermediate zone; we thus have—



$$a = b = c;\ \xi = \eta = \zeta = 90°$$

The five elements are determined.

The most general form, $h\,k\,l$, consists of forty-eight faces, whose distribution is shown in Fig. 26.

In the previous development only the most general form, $h\,k\,l$, has been considered; it is, however, very easy by specializing the symbols, as by an equation of two indices, for instance, or by conditions which can be conceived in the projection, to represent all the forms of a system by the number and signs of the faces.

We wish, for example, the symbol of the faces of the six-sided, the twelve-faced pyramid of the hexagonal system. Their symbols result from the relation of the zones. On the other hand, a simple inspection of the planes of symmetry of this system shows that a face occurring in the zone [(1 1 1) (2 $\bar{1}$ $\bar{1}$)] has on the upper side five similar faces. Thus it results that the symbol of the opposite rhombohedron, similar to the primitive rhombohedron, is ($\bar{1}$ 2 2), according to the formula, (p. 22.) Partial forms have not been included in the above representation, any more than the researches on the symmetry of lines and planes, which will be given in another place.

SECTION III.

OPTICAL RELATIONS OF CRYSTALS.

§ 1.—DOUBLE REFRACTION AND ABSORPTION.

It is known that in media of equal density throughout, also in uncrystallized media, a ray of light moves in every direction with the preservation of its condition of vibration; that, further, its velocity of propagation

is only dependent on the color of the beam of light, and on a factor which is constant for the entire medium, and not from the direction in which it moves.

If, therefore, a beam of light, under any condition of vibration, enters such an isotrope medium, it can very easily, taking into consideration its angle of incidence, change the direction, and, taking into consideration its color and molecular constants, the velocity of its propagation; the condition of vibration, however, remains constant. The condition of vibration of the beam of light is said to be completely polarized, partially polarized, or unpolarized, according as the whole of the light or only part of it vibrates in a constant path, or this course takes in an infinitely short space every possible transversal position. In the first case, where the whole light has a constant path of vibration, we say again that the light is polarized in a straight line, circularly, or elliptically, according as the path of oscillation is a straight line, perpendicular to the direction of propagation, a circle, or an ellipse. The movement of the light in an isotrope medium is, therefore, dependent on that of the incident light, the angle of incidence, and a molecular constant.

In a crystallized medium, in which the density can be supposed variable with the direction, only two beams of light of a determined velocity of propagation for each color, and determined direction of vibration, can, in general, be propagated in any determined direction; on the contrary, a beam of light entering a crystalline medium will not only be deviated from its direction, but separated into two divergent beams, each one of which, according to its direction in the crystal, will have variable velocities of propagation and direction of vibration.

Just as the intensity of the light is weakened by its passage through an isotrope medium, and has different strengths for different colors, so is it the case with crystalline media, only here the unequal absorption for different colors depends on the direction in the crystal; the same is here true as of the manner of vibration and the velocity of propagation. The same direction in a crystal corresponds thus to two determined beams with determined velocities of propagation, direction of vibration, and absorption; and a ray of light entering a crystal is divided into two beams of determined but different directions of propagation, velocities, directions of vibration, and absorption.

§ 2.—The Ellipsoid of Polarization.

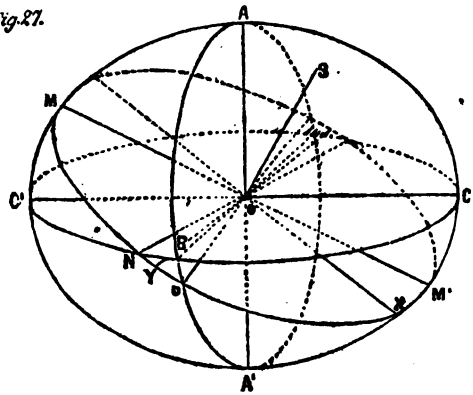
The law according to which the whole movement of light in a crystal is determined can, so far as is necessary for our present purpose, be enunciated as follows:

In every crystal an ellipsoid with three axes can be constructed in such a way that the velocity of propagation and the direction of vibration of the two rays of light, which can move in a fixed direction in a crystal, may be determined by the major and minor axes of the ellipse, which are formed when, from the center of the ellipsoid, a plane is passed

perpendicular to the given direction of propagation of both of the beams, and this prolonged to its section with the ellipse.

Let o A, o B, o C, (Fig. 27,) be the principal axes of the ellipsoid, at right angles to each other; S o, the direction, passing through the center, in which the two beams of light should move. Let us pass through O a plane perpendicular to o S, which cuts the ellipsoid in the points M N o M′, which points belong to an ellipse whose major and minor half-axes are o X and o Y; of these two beams, propagated in the direction S o, the one has the direction of vibration o X and the velocity of propagation $\frac{1}{0}$, and the other O Y and $\frac{1}{0}$.

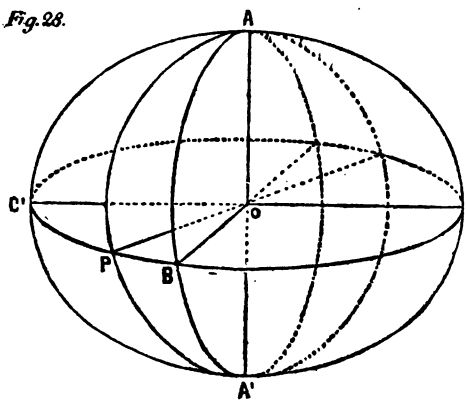
Fig. 27.



The situation and the length of the principal axis of this ellipsoid are, in general, different for every color. The absorption of the light in any direction can also be determined from the principal axis. With the co-efficient of absorption of the principal axis we can again construct an ellipsoid whose axes correspond to those of the ellipsoid of polarization. The co-efficient of absorption for the two rays of light corresponding to a direction will be determined sometimes by the ellipse-section and sometimes by the absorption-ellipsoid; the major and minor axes of this ellipse, it is true, do not coincide exactly, but they do approximatively with those of the direction of vibration.

In the most general case, which we shall first discuss, the three axes of the ellipsoid are of unequal lengths; they will be called axes of polarization or of elasticity; by the last is also specially understood their reciprocal lengths, as—

Fig. 28.



$$a = \frac{1}{o\,A}; \quad b = \frac{1}{o\,B}; \quad c = \frac{1}{o\,C}$$

in which $a > b > c$ is chosen; hence the distances o A, o B, o C, are themselves proportional to the principal quotient of refraction.

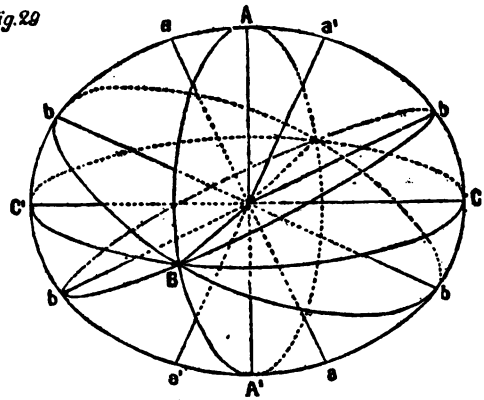
A plane of the axes containing two axes of elasticity is called the principal section, and is perpendicular to the third axis.

A plane parallel to one axis, as o C, (Fig. 28,) cuts the ellipsoid in an

ellipse, C P C′, one axis of which coincides with the known axis of elasticity, and the other axis, *o* P, is perpendicular to it in the principal section A B *o*.

A plane M N *o*, (Fig. 27,) inclined to all three of the axes of elasticity, cuts the ellipsoid in an ellipse, whose axes are not parallel to any of the axes of elasticity. In general, $a > b > c$ is true in the principal section A *o* C, (Fig. 29,) whenever there is a radius, O *b*, whose length is equal to the middle axis of elasticity, O B. If a plane, B *o b*, is passed through this last and this radius, their section of the ellipsoid is a circle; the normal *o a* to this circle-face lies in the principal section of the largest and smallest axes of elasticity, *o* A C, and is called an *optical axis*. This ellipsoid, which has three axes, has two optical axes, *o a* and *o a′*, (Fig. 29,) which are in the planes of the greatest and smallest axes of elasticity, and are situated symmetrically with regard to both.

Fig. 29



The optical axes form with each other two supplementary angles, an acute, 2 Y *a*, and an obtuse, 2 Y *o*, so that 2 Y $a = 180° - 2$ Y *o*, which are equally divided by the axes A and C; that axis which divides the acute-angle axis is called the *first middle line, (bisectrix,)* and the one which divides the obtuse-angle axis is called the *second middle line*, so that two cases are again possible:

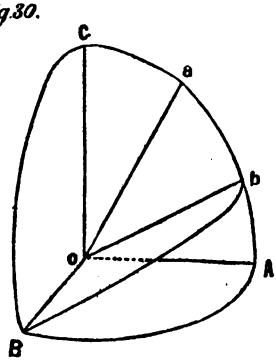
First middle line $\frac{a}{-}$ second middle line *c*: negative crystal.

First middle line $\frac{c}{+}$ second middle line *a*: positive crystal.

The first case is assumed in Fig. 29. Fig. 30 shows a sketch of the last.

According as the nature of double refraction consists in a difference of the velocity of propagation and of the direction of vibration of the two beams of light capable of being propagated in the same direction, it is at once clear that the double refraction must disappear along the optical axes. The plane normal to a beam of light, that is, the one which propagates itself in the direction of an optical axis, cuts the ellipsoid in a circle; the velocities of propagation of the beams of light given by two radii are equal to each other; the directions of vibration are undetermined, *i. e.*, remain unchanged, as they were before their entrance into the crystalline medium.

Fig. 30.



If a crystal has a plane of symmetry, it must coincide with a principal section of the ellipsoid for every color, because an ellipsoid with three axes is symmetrical only in its principal sections; this coincidence must not, however, occur in the same principal section for every color; thus, for red light, $b\,c$, and for blue light, $a\,c$, may fall in the plane of symmetry. If two axes of elasticity of the same ellipsoid are equal, their principal section will be a circle, and the two axes become reduced to one; if, for instance, the third axis of elasticity is perpendicular to this principal section, the ellipsoid is an ellipsoid of rotation. The sections of such an ellipsoid, with a plane, are either *perpendicular* to the optical axis, section a circle, no double refraction, direction of vibration undetermined; or *parallel* to the optical axis, section an ellipse, one axis of which is the optical axis, the other has a constant value, which is that of the axis of elasticity originating in the circle; or *inclined* to the optical axis, section an ellipse, whose axes are inclined to the optical axes. Ellipsoids with a single axis are of two kinds, lengthened or flattened, according as—

$b = c$; $\frac{a}{-}$ the optical axis; negative crystal, (Fig. 31.)

$a = b$; $\frac{c}{+}$ the optical axis; positive crystal, (Fig. 32.)

Fig. 31.

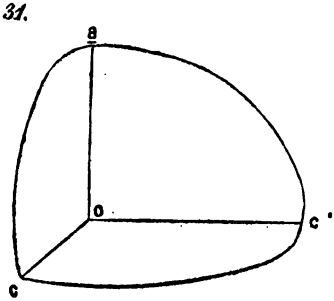
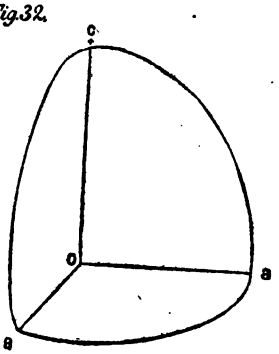


Fig. 32.



If all three of the axes of elasticity of the ellipsoid are equal to each other, it becomes a sphere; every section by a plane will be a circle; all the axes of such a circle will be equal to each other. Such a crystal is monorefringent, and has no determined direction of vibration, that is to say, the direction of vibration of the beam of light entering the crystal remains the same.

As has been already mentioned above, the relations of absorption in the whole crystal can be determined if they are given for the three axes of elasticity. If we construct an ellipsoid from the three principal absorption-constants (for a determined color) as axes, we find, exactly as in the ellipsoid of polarization, the amount of absorption for a given direction in the crystal by passing a normal plane and determining the axes of the ellipse-section so produced.

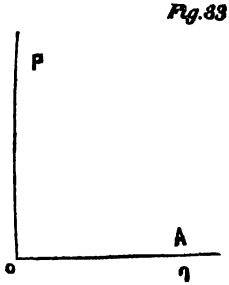
§ 3.—Optical Relations of Plane Plates whose Sides are Parallel.

We shall first consider the relations of plane plates of crystals whose sides are parallel, in straight-lined parallel light and perpendicular incidence. Let the entering beam of parallel straight-lined light polarized by any means, such as a nichol prism, heropathite, or a plate of tourmaline, fall perpendicularly upon a plane plate of the crystal whose sides are parallel. In consequence of its perpendicular incidence, for we can treat parallel light always so, the beam of light enters the crystal without deviation; in this defined direction only two beams, whose direction of vibration is determined according to § 2 of this section, can transmit themselves in the crystal, because we bring the plane of the plates, which is perpendicular to the path of the beam of light, into the section of the ellipsoid of polarization.

The entering beam of light must now be divided, according to these two lines at right angles to each other, into two component parts, which then follow the same path entirely through the crystal; passing out of it, however, they fall upon a second polarizing arrangement, the analyzer, which, as the polarizer, allows the vibrating light to pass only in a given direction. Here the two beams of light are divided in such a way that only that component which falls in the plane of vibration of the analyzer comes out of it; finally, both these components, polarized in straight lines, have similar directions of vibration, and the same path, and for this reason unite in a straight polarized beam of light, with the same direction of vibration as component and analyzer.

We suppose that both the polarizer and the analyzer are so placed that their directions of vibration are parallel to each other, which position is once for all determined. Let us now turn the crystal-plate in its own plane until its directions of vibration come together, the one, $o\xi$, (Fig. 33,) with o P of the polarizer, the other, $o\eta$, with o A of the analyzer, and we have the following result:

Fig. 33.



Straight-lined polarized light comes from the polarizer in the direction of vibrating light, o P. By its entrance into the crystal it will be divided in the direction $o\xi$ and $o\eta$, which is its direction of vibration; thus no component escapes, especially in the direction $o\eta$, but the beam passes through the plate in the direction $o\xi$, and passes out of it with the direction of vibration $o\xi$, falls upon the analyzer, is here divided into two components, of which only the parallel one, o A, is allowed to pass parallel to o A; however, $o\xi$ gives out no component, which means that in this case no light whatever comes through the analyzer.

We see also that any crystal-plate with parallel planes appears dark when placed between polarizers which are at right angles to each other,

as soon as its direction of vibration coincides with that of the analyzer and polarizer.

In order to observe the absorption, we have only, when the plate is in the position of darkness, to take away the analyzer or the polarizer. In this case only the color corresponding to the one direction of vibration of the plate appears. This occurrence thus shows itself entirely analogous to the previous one.

It is at once clear that a plate perpendicular to an optical axis appears dark in every position of the crossed polarizers. Respecting the relations of a plate with parallel sides between polarizers in a cone of monochromatic light, we only remark that optical axes are shown by a system of very nearly concentric rings, through whose center a dark, straight or hyperbolic beam, or a dark cross, appears. The appearance of these in white light will be described for some of the systems.

§ 4.—Optical Relations in each Crystalline System.

As has already been mentioned above, the position of the principal optical section and the value of the axes of elasticity are different for different colors. A coincidence takes place only in the case of the existence of one or more planes of symmetry, because such a one must always be a principal optical section.

1. Triclinic system.—No plane of symmetry. The position of the ellipsoid of polarization for the different colors cannot be determined *a priori;* the axes of elasticity are inclined to the axes of the crystal; all the principal optical sections are dispersed, that is, have a different position for every color. In general, the dispersion of the principal section, both here and in the following crystalline systems, is small, and seldom goes beyond one or two degrees. The appearances of color in plane plates with parallel sides, which allow the optical axes to be distinguished, are in monochromatic light as follows: A plate perpendicular to the bisectrix shows, when the polarizers are crossed, a black cross, (Fig. 34,) upon one arm of which the elliptical rings of the optical axes

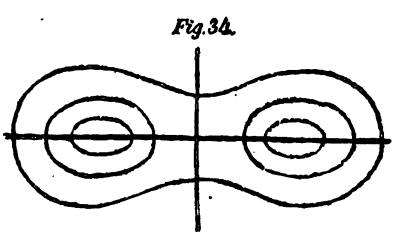


Fig. 34.

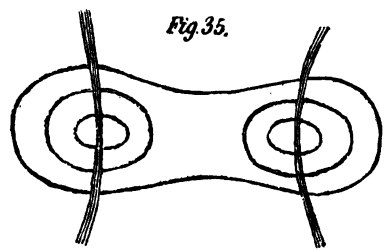


Fig. 35.

appear surrounded by lemniscates if the principal axes of the plate coincide with those of the polarizer; when hyperbolæ (Fig. 35) pass through the rings of the axes, the principal sections of the plate are inclined 45° to the polarizer; in white light the rings of some colors

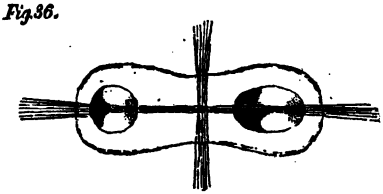
appear superposed; on account of the dispersion of the principal section, both of the two images of the axes, and also the arrangement of the colors in both of them, will be unsymmetrical with regard to the principal section, which is marked by a somewhat faint and black beam. The detail of this image of the axes is most simply described by saying that a union of the cases of dispersion, met with in variable intensity in the following system of crystallization, is to be here observed.

2. MONOCLINIC SYSTEM.—One plane of symmetry. A principal optical section of every color must coincide with the plane of symmetry, so an axis of elasticity of every color must coincide with the axes of the crystal *o* Y, perpendicular to the plane of symmetry. The two other principal sections, as also the two axes of elasticity lying in the plane of symmetry, are dispersive for the different colors. There are here three possible cases:

First. The principal section *a c*, containing the optical axes of a color, coincides with the plane of symmetry, *inclined dispersion*, (*dispersion inclinée* of Descloiseaux.) The general case is, that the analogous principal sections have for all colors very nearly the same position; in this case the optical axes, for all the colors, lie in the plane of symmetry; the image of a plate perpendicular to a bisectrix, (convergent light,) on account of the correspondence of the direction of vibration of the plate and the polarizer, is symmetrical with respect to the black beam joining the image of the axes, (Fig. 36.)

Secondly. The principal section of the axes is perpendicular to the plane of symmetry; the bisectrix lies in the plane of symmetry, *horizontal dispersion*, (*dispersion horizontale* of Descloiseaux.) In this case *c b* for positive crystals, and *a b* for negative crystals, coincide with the plane of symmetry.

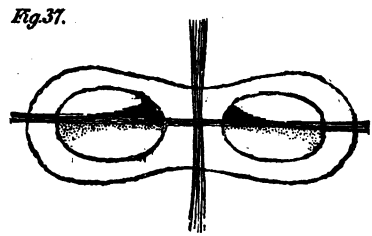
Fig. 36.



If the general case of the approximate coincidence of similar principal sections for different colors is selected, we see that here the planes of the optical axes are dispersive. The image of the axis appears symmetrical with respect to a beam perpendicular to the line of the optical axes, (Fig. 37.)

Thirdly. The section of the axes *a c* and the bisectrix are perpendicular to the plane of symmetry; the principal section *a b* for positive, and *c b* for negative crystals, coincide, therefore, with the plane of symmetry; *cross-wise dispersion*, (*dispersion croisée* of Descloiseaux.) The planes of the axes are dispersive.

Fig. 37.

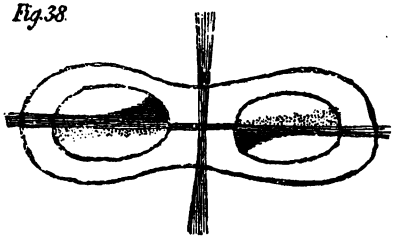


Under the same supposition as before, the image of the axes will not be symmetrical with regard to any line;

the planes of the axes appear round the normal to the plate, (second crystallographic axes, *o* Y, bisecting,) dispersed in the shape of a fan, (Fig. 38.)

3. ORTHORHOMBIC SYSTEM.—Three unequal planes of symmetry at right angles to each other. Every plane of symmetry must coincide with a principal section; here the position of the principal optical section is completely determined, and only the value and position of the axes of elasticity are undetermined. In most cases the similar principal sections of all colors coincide, as also do the axes of elasticity *a*, *b*, *c*.

Fig. 38.



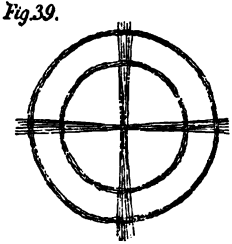
The image of the axes, according to the former suppositions, is symmetrical with regard to the two black beams; it appears also in white light, similar to Fig. 34, but in this case the black ellipses are replaced with color. The principal optical section is not dispersive; the optical axes, however, are; that is, the angle of the axes is different for different colors, as in both the previous systems.

4. RHOMBOHEDRIC SYSTEM.—Three tautogonal and similar planes of symmetry, inclined at an angle of 60°. Every one of these must be a principal section of the ellipsoid; this is only possible if all these zones belonging to the section of the ellipsoid are equal to each other; that is, it is an ellipsoid of rotation; the principal section perpendicular to the plane of symmetry is a circle; the axis of the zone of symmetry is the optical axis of all the colors. Here, as we have already mentioned, two cases are possible, positive or negative crystals, according as $b = c$ or $a = b$.

If we again make the supposition that the similar axes of elasticity coincide for all colors, we get, as the image of a plate cut perpendicular to the optical axis between two crossed polarizers, a black cross with concentric colored rings, (Fig. 39.)

5. TETRAGONAL SYSTEM.—Five planes of symmetry, four of which are inclined 45° to each other, every alternate one being similar, the fifth perpendicular to the four others.

Fig. 39.



A principal optical section is parallel to this last, as the hypothetical plane of symmetry 0 0 1. All its perpendicular ellipsoid sections must be equal to each other, because in this zone four planes of symmetry exist, all of which must be principal sections of the ellipsoid. The tetragonal system, therefore, is optically exactly like the rhombohedral.

6. HEXAGONAL SYSTEM.—Seven planes of symmetry, six tautogonal inclined 30°, every alternate three similar, one perpendicular to them. This last, taken as a principal section, makes, as in the two previous

systems, all sections perpendicular to it similar, on account of the symmetry according to the six tautozonal planes of symmetry; on this account, therefore, having the same optical relations. The base 111 is perpendicular to the optical axis.

7. TESSERAL (ISOMETRIC) SYSTEM.—Nine planes of symmetry, three perpendicular to each other and similar, the other six intercalated tautozonally at 45° between each two of the first.

If we take the first three planes of symmetry parallel to the three principal sections, it results immediately, from the existence of the other planes of symmetry, that the ellipsoid of polarization must be a sphere whose radius is different for different colors. A sphere has only circular sections; therefore, simple refraction is produced in all directions.

We have above considered only the cases where the similar principal sections of all colors very nearly coincided; the exceptions to this law are really very rare, and present no difficulties. Observation by means of monochromatic glasses or sources of light always allows a very quick orientation.

We have also in the above description left out crystals with one axis, which polarize circularly, because they, in spite of the greatest theoretical differences, can practically be regarded exactly as the other monoaxial crystals, with the exception of the image of the axis, which inside of the rings shows that the black cross is replaced by a uniform color, which is dependent on the thickness of the plate.

It is now no longer necessary to describe the special behavior of sections of different crystals with respect to the orientation of their direction of vibration. The orientation of the ellipsoid, with regard to the axes of the crystal and their respective planes of symmetry, is given above; if, therefore, the crystallographic orientation of a plate is known, the kind of section in the ellipsoid and the directions of vibration can be at once determined. Inversely, the experimentally easily-determined position of the direction of vibration of a section of known crystallographic orientation gives a starting-point for the determination of the system.

Reviewing the method of development of the foregoing sketch, we see, as the starting-point, the law of experience, that by the selection of a certain method of representation, the symbols of all faces and zones consist of whole numbers, whose relations with one another are therefore rational numbers.

From the rationality of these numbers follows, in a way which we could only briefly dwell upon, that only such groupings of faces are possible which belong to one of the seven different kinds of symmetry, the seven systems of crystallization. From the general law of the

movement of light in crystals resulted the ellipsoid of polarization for the derivation of all special rules. The relations of symmetry of the separate systems of crystallization allow us to discover in a very simple manner the nature of the ellipsoid of polarization, and with it the optical characters of every system, with which we have completed the object of this memoir.